APPLICABILITY OF THE ST. VENANT PRINCIPLE IN THE THEORY OF THIN ELASTIC ENVELOPES UNDER THE EFFECT OF FORCE AND TEMPERATURE

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The stressed-and-strained state of the most unfavorable representatives of envelopes of zero Gaussian curvature from the point of view of applicability of the St. Venant principle – the cylindrical ones – under forces and temperature fields with piecewise-smooth variation along the genetratrix and cosinusoidal variation along the contour is investigated. Results of an experimental investigation of the envelopes are presented.

It is known that the so-called St. Venant principle plays an important role in the theory of elasticity and the resistance of materials [1]. According to this principle, tensions caused by an external load balanced in its cross section have a local character and damp quickly with distance. However, as is noted [2], "the St. Venant principle has a limited area of application not only for thin-walled rods of open unstrained contour but also for thin-walled rods and envelopes of closed contour, if they are not strengthened by stiffening." Up to now there is no strict proof of the St. Venant principle, and attempts to build it are dedicated mostly to estimation of the error as applicable to prismatic bodies. As concerns rods and envelopes, here a pleasant exception is just the case of axisymmetric strain of the latter, when the end effect appears [3, 4]. In it the bending strain damps rapidly with distance from the distortion lines, and it practically disappears at a distance of $(2.5–5)\sqrt{Rh}$, for example, from the boundary of the loaded region. At the same time one often encounters a free interpretation of possible areas of application of the St. Venant principle even in the cases where arbitrary loads affect large-scale thin-walled constructions, reaching equilibrium not on the thickness of the envelope, where this principle is true, but on the contour of the thin-wall construction or on the contour of large cuts, where its inapplicability should not be doubted. This stimulated our investigation of the problem, with cylindrical envelopes, in which the most deeply penetrating stressed-and-strained state occurs, being taken as the object of investigation. This means that such envelopes are the most unfavorable from the viewpoint of the applicability of the St. Venant principle, if envelopes of negative Gaussian curvature, which have not found wide practical use, are not taken in consideration. Envelopes of finite length affected by longitudinal and radial loads as well as temperature fields varying in a piecewise-continuous manner along the generatrix and cosinusoidally around the circumference are considered. The envelopes have combinations of the following boundary conditions: a free edge, a hinge-fixed edge (an edge with a diaphragm that is rigid in its plane), and a rigidly pinched edge. Numerical information on the stressed-and-strained state of the most important points of the envelopes is obtained on the basis of the solutions of the boundary-value problems built.

1. In using the theory of thin elastic envelopes built on the basis of adoption of the Kirchhoff–Love hypotheses, the problem of the action of longitudinal and radial surface loads $p_1(\alpha, \beta)$, $p_2(\alpha, \beta)$ on a circular cylindrical envelope, a temperature field $t^*(\alpha, \beta)$ that is constant over the thickness of the envelope, and a temperature difference over the thickness $t^{**}(\alpha, \beta)$ can be reduced to one resolving partial differential equation with respect to the resolving function $\Phi(\alpha, \beta)$ [5, 6]:

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$$\Phi(\alpha,\beta) = -\frac{R^4}{D} p_1(\alpha,\beta) + \frac{R^4}{D} p_2(\alpha,\beta) +$$

$$+ \frac{1+\nu}{c^2} \alpha_t R t^*(\alpha,\beta) - \frac{1+\nu}{6c^2} \alpha_t h t^{**}(\alpha,\beta), \tag{1}$$

$$= \nabla^4 (\nabla^2 + 1)^2 - 2(1-\nu)\left(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\nabla^2 +$$

$$+ \frac{1-\nu^2}{c^2}\frac{\partial^4}{\partial\alpha^4}; \quad \nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}, \quad c^2 = \frac{h^2}{12R^2},$$

where $2t^* = t_1 + t_2$, $2t^{**} = t_2 - t_1$.

Displacements, internal forces, bending moments, etc. are expressed in terms of the resolving function by corresponding differential relations that are different under the effect of force and temperature $p_1(\alpha,\beta)$, $p_2(\alpha,\beta)$, $t^*(\alpha,\beta)$, $t^{**}(\alpha,\beta)$ mentioned above [5, 6].

The above-mentioned boundary conditions can be formulated in the following manner:

$$T_1 = S = Q_1^* = G_1^* = 0 \quad \text{(free edge)}, \tag{2}$$

$$T_1 = G_1 = v = w = 0 \quad \text{(hinge edge)}, \tag{3}$$

$$u = v = w = w'_\alpha = 0 \quad \text{(rigidly pinched edge)}. \tag{4}$$

Here, u, v, w, T_1, S, Q_1^*, G_1 are the longitudinal, cirumferential, and radial displacements, the longitudinal normal force, the shearing force, the cutting force, and the longitudinal bending moment, respectively.

2. First we dwell on the problem of the action of a longitudinal force on envelopes of finite length. We express the former in the following form:

$$p(\alpha,\beta) = p_0 \theta(\alpha) \cos n\beta \quad (n = 0, 1, 2, 3, \ldots) \tag{5}$$

(index 1 on the longitudinal load is omitted hereinafter), where p_0, $\theta(\alpha)$ are the amplitude value of the load and the dimensionless piecewise-continuous (step in a particular case) function of its distribution along the generatrix of the envelope.

The resolving function $\Phi(\alpha,\beta)$ and the sought factors are expressed in the following form:

$$\Phi(\alpha,\beta) = \Phi_n(\alpha) \cos n\beta, \tag{6}$$

$$\begin{gathered} u(\alpha,\beta) = U_n(\alpha)\cos n\beta, \quad v(\alpha,\beta) = V_n(\alpha)\sin n\beta, \\ w(\alpha,\beta) = W_n(\alpha)\cos n\beta, \quad T_1(\alpha,\beta) = T_{1n}(\alpha)\cos n\beta, \\ S(\alpha,\beta) = S_n(\alpha)\sin n\beta, \quad G_2(\alpha,\beta) = G_{2n}(\alpha)\cos n\beta. \end{gathered} \tag{7}$$

After substituting (5), (6) in (1) we arrive at an ordinary differential equation with respect to the function $\Phi_n(\alpha)$:

$$_n\Phi_n(\alpha) = -(R^4/D)\, p_0\, \theta(\alpha), \tag{8}$$

$$_n(\ ) = \left(\frac{d^2}{d\alpha^2} - n^2 + 1\right)^2 \left(\frac{d^2}{d\alpha^2} - n^2\right)^2 - 2(1-\nu)\frac{d^2}{d\alpha^2}\left(\frac{d^2}{d\alpha^2} - n^4\right) + \frac{1-\nu^2}{c^2}\frac{d^4}{d\alpha^4}.$$

Integrating Eq. (8), we obtain the following solution:

$$\Phi_n(\alpha) = C_1 \exp(-r_{1n}\alpha)\sin\kappa_{1n}\alpha + C_2 \exp(-r_{1n}\alpha)\cos\kappa_{1n}\alpha +$$
$$+ C_3 \exp(-r_{2n}\alpha)\sin\kappa_{2n}\alpha + C_4 \exp(-r_{2n}\alpha)\cos\kappa_{2n}\alpha +$$
$$+ C_5 \exp r_{1n}\alpha \sin\kappa_{1n}\alpha + C_6 \exp r_{1n}\alpha\cos\kappa_{1n}\alpha +$$
$$+ C_7 \exp r_{2n}\alpha\sin\kappa_{2n}\alpha + C_8 \exp r_{2n}\alpha\cos\kappa_{2n}\alpha + \hat{\Phi}_n(\alpha), \tag{9}$$

where $C_1, C_2, C_3, \ldots, C_8$ are arbitrary constants determined from two combinations of the boundary-value conditions (2)-(4); r_{1n}, κ_{1n}, r_{2n}, and κ_{2n} are the real and imaginary parts of the eight complex roots $\lambda_{1-4} = \pm r_{1n} \pm i\kappa_{1n}$, $\lambda_{5-8} = \pm r_{2n} \pm i\kappa_{2n}$ of the following characteristic equation:

$$(\lambda^2 - n^2 + 1)^2(\lambda^2 - n^2)^2 - 2(1-\nu)\lambda^2(\lambda^4 - n^4) + (1-\nu^2)c^{-2}\lambda^4 = 0. \tag{10}$$

In expression (9) the particular solution corresponding to the right-hand side of Eq. (8) is expressed as $\hat{\Phi}_n(\alpha)$.

In relation (5) the numbers n of the harmonics take the following values: $n = 0$, $n = 1$, $n = 2$, $n = 3$, ... And when it goes to a discussion on the applicability of the St. Venant principle, it is natural to investigate the behavior of a stressed-and-strained state that changes along the generatrix most slowly. This is the stressed-and-strained state that corresponds to the harmonic number $n = 2$, since it changes along the envelope's generatrix most slowly. The cases $n = 0$, $n = 1$, which correspond to an axisymmetric state under tension-compression of a rod ($n = 0$) and bending of an envelope without contour strain as a beam ($n = 1$), are not discussed here, for they are elementary and have been well investigated.

Let us consider the numerical values of the roots of the charactersitic equation (10) that correspond to a cosinusoidal effect $p_0 \cos 2\beta$ on an envelope for which $R/h = 100$. They are presented in Table 1, from which it is seen that for $n = 2$ the roots are clearly divided into large and small [6]. Precisely the small roots attract our attention in investigation of the problem posed, since it is just they who characterize a slowly damping stressed-and-strained state of the envelope. Let us pay attention to one fact: the small roots in Table 1 practically coincide with the roots of the characteristic equation of the semi-zero-moment theory of envelopes, which can easily be obtained from the characteristic eguation of the general theory of envelopes presented in the form of (10). In the case of the semi-zero-moment theory the characteristic equation is

$$\lambda^4 + 4\mu_n^4 = 0, \quad 4\mu_n^4 = c^2(1-\nu^2)^{-1} n^4 (n^2 - 1)^2, \tag{11}$$

and its roots are $\lambda_{1-4} = (\pm 1 \pm i)\mu_n$.

For the harmonic number $n = 2$ at $R/h = 100$ we have $\mu_2 = 0.135$. The difference between the values of the root according to the general theory and the semi-zero-moment theory does not exceed several percent. Therefore for further investigation we shall use the relations of the semi-zero-moment theory of envelopes, for which the resolving equation takes the form

$$\mathscr{L}\Phi(\alpha,\beta) = -(R^2/Eh)\,p(\alpha,\beta), \tag{12}$$

$$(\mathscr{L}) = \frac{\partial^4}{\partial\alpha^4} + \frac{c^2}{1-\nu^2}\frac{\partial^4}{\partial\beta^4}\left(\frac{\partial^2}{\partial\beta^2} + 1\right)^2.$$

TABLE 1. Real and Imaginary Parts of the Roots of the Characteristic Equation (10) for $n = 2$

r_{1n}	κ_{1n}	r_{2n}	κ_{2n}
0.1359	0.1335	13.01	12.70

TABLE 2. A Solution for the Case of a Longitudinal Load $q_0 \cos n\beta$ Concentrated on the Line $\alpha = \xi$

Amplitude values of the forces and displacements	$T_{1n}(0)$	$S_n(0)$	$V_n^*(0)$	$U_n^*(0)$	q_0
$T_{1n}(\alpha)$	$\Phi_2(\alpha)$	$-\frac{n}{2\mu_n}[\Phi_1(\alpha)+ + \Phi_3(\alpha)]$	$-\frac{2\mu_n^2}{n}\Phi_4(\alpha)$	$\mu_n[\Phi_3(\alpha) - - \Phi_1(\alpha)]$	$-K_{TT}(\alpha - \xi)$
$S_n(\alpha)$	$-\frac{\mu_n}{n}[\Phi_3(\alpha) - \Phi_1(\alpha)]$	$\Phi_2(\alpha)$	$\frac{2\mu_n^3}{n^2}[\Phi_1(\alpha)+ \Phi_3(\alpha)]$	$\frac{2\mu_n^2}{n}\Phi_4(\alpha)$	$-K_{ST}(\alpha - \xi)$
$V_n^*(\alpha)$	$\frac{n}{2\mu_n^2}\Phi_4(\alpha)$	$\frac{n^2}{4\mu_n^3}[\Phi_3(\alpha)- -\Phi_1(\alpha)]$	$\Phi_2(\alpha)$	$\frac{n}{2\mu_n}[\Phi_1(\alpha)+ \Phi_3(\alpha)]$	$-K_{VT}(\alpha - \xi)$
$U_n^*(\alpha)$	$\frac{1}{2\mu_n}[\Phi_1(\alpha)+ \Phi_3(\alpha)]$	$-\frac{n}{2\mu_n^2}\Phi_4(\alpha)$	$\frac{\mu_n}{n}[\Phi_3(\alpha) - \Phi_1(\alpha)]$	$\Phi_2(\alpha)$	$-K_{UT}(\alpha - \xi)$

Note: $\Phi_1(\alpha) = \cosh \mu_n\alpha \sin \mu_n\alpha$, $\Phi_2(\alpha) = \cosh \mu_n\alpha \cos \mu_n\alpha$, $\Phi_3(\alpha) = \sinh \mu_n\alpha \cos \mu_n\alpha$, $\Phi_4(\alpha) = \sinh \mu_n\alpha \times \sin \mu_n\alpha$, $K_{TT}(\alpha - \xi) = \Phi_2(\alpha - \xi)$, $K_{ST}(\alpha - \xi) = -(\mu_n/n)[\Phi_3(\alpha - \xi) - \Phi_1(\alpha - \xi)]$, $K_{VT}(\alpha - \xi) = (n/2\mu_n^2)\Phi_4(\alpha - \xi)$, $K_{UT}(\alpha - \xi) = (1/2\mu_n)[\Phi_1(\alpha - \xi) + \Phi_3(\alpha - \xi)]$.

The resolving function $\Phi(\alpha, \beta)$ and the sought power and strain factors $u(\alpha, \beta)$, ..., $G_2(\alpha, \beta)$ can be expressed in the form of (6) and (7).

The solution for a piecewise-continuous and step load along the envelop's generatrix is obtained on the basis of the solution for a load concentrated in the longitudinal direction, written in the form of Table 2. To build it the method of initial parameters [5, 6] is used. The particular solution that corresponds to a load varying arbitrarily on the segment $[\xi_1, \xi_2]$ is presented in the next to last column of Table 3, and the particular solution for a piecewise-continuous load is given in the last column. The general solution of the resolving equation of the problem is presented in the left part of Table 2. To shorten the notation, the influence functions are written as $K_{TT}(\alpha)$, ..., $K_{UU}(\alpha)$ in Table 3. As is known, the initial parameters $T_{1n}(0)$, $S_n(0)$, $V_n^*(0)$, $U_n^*(0)$, playing the role of arbitrary constants, can be determined from the condition of satisfying the tangential boundary conditions from (2)-(4), to which the factors whose amplitude values are in the left-hand sides of Tables 2 and 3 relate. The length of the loaded region is $2\alpha_0 R$, ξ is the dimensionless coordinate of its center; $\xi_1 = \xi - \alpha_0$, $\xi_2 = \xi + \alpha_0$, where ξ_1 and ξ_2 are the dimensionless coordinates of the boundary of the loaded region. The influence functions in the left part of Table 3 differ from zero when $\alpha - \xi_1 \geq 0$, $\alpha - \xi_2 \geq 0$.

After determination of the initial parameters and amplitude quantities $U_n(\alpha)$, $V_n(\alpha)$, $T_{1n}(\alpha)$, $S_n(\alpha)$ satisfying the formulated boundary-value problem, displacements, forces, and bending moments can be written in form of (7). For example, the longitudinal displacement and the normal force can be written as

$$u(\alpha, \beta) = U_0(\alpha) + U_1(\alpha)\cos\beta + U_2(\alpha)\cos 2\beta + U_3(\alpha)\cos 3\beta + \ldots$$
$$T_1(\alpha, \beta) = T_{10}(\alpha) + T_{11}(\alpha)\cos\beta + T_{12}(\alpha)\cos 2\beta + T_{13}(\alpha)\cos 3\beta + \ldots \quad (13)$$

TABLE 3. A Solution for the Case of the Action of a Longitudinal Load $p_0 \cos n\beta$ That Is Arbitrary and Piecewise-Continuous on the Section $[\xi_1, \xi_2]$

Amplitude values of the forces and displacements	$T_{1n}(0)$	$S_n(0)$	$V_n^*(0)$	$U_n^*(0)$	p_0R	
$T_{1n}(\alpha)$	$K_{TT}(\alpha)$	$K_{TS}(\alpha)$	$K_{TV}(\alpha)$	$K_{TU}(\alpha)$	$-\int_{\xi_1}^{\xi_2} \Phi_2(\alpha-\xi)\theta(\xi)d\xi$	$\frac{1}{2\mu_n}[\Phi_1(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) + \Phi_3(\alpha-\xi_2) - \Phi_4(\alpha-\xi_1)]$
$S_n(\alpha)$	$K_{ST}(\alpha)$	$K_{SS}(\alpha)$	$K_{SV}(\alpha)$	$K_{SU}(\alpha)$	$\frac{\mu_n}{n}\int_{\xi_1}^{\xi_2} [\Phi_3(\alpha-\xi) - \Phi_1(\alpha-\xi)]\theta(\xi)d\xi$	$\frac{1}{n}[\Phi_2(\alpha-\xi_1) - \Phi_2(\alpha-\xi_2)]$
$V_n^*(\alpha)$	$K_{VT}(\alpha)$	$K_{VS}(\alpha)$	$K_{VV}(\alpha)$	$K_{VU}(\alpha)$	$-\frac{n}{2\mu_n^2}\int_{\xi_1}^{\xi_2} \Phi_4(\alpha-\xi)\theta(\xi)d\xi$	$\frac{n}{4\mu_n^3}[\Phi_1(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) - \Phi_3(\alpha-\xi_2) + \Phi_3(\alpha-\xi_1)$
$U_n^*(\alpha)$	$K_{UT}(\alpha)$	$K_{US}(\alpha)$	$K_{UV}(\alpha)$	$K_{UU}(\alpha)$	$-\frac{1}{2\mu_n}\int_{\xi_1}^{\xi_2} [\Phi_1(\alpha-\xi) + \Phi_3(\alpha-\xi)\theta(\xi)d\xi$	$\frac{1}{2\mu_n^2}[\Phi_4(\alpha-\xi_2) - \Phi_4(\alpha-\xi_1)]$

In (13) the first two terms ($n = 0$, $n = 1$) correspond to a situation in which the contour of the envelope is not strained and the envelope behaves like a rod or a beam. All other terms ($n \geq 2$) reflect a stressed-and-strained state accompanied by strain of the contour of the envelope's cross section.

Here we shall give final analytical expressions for the sought factors of the stressed-and-strained state of an envelope having one edge hinge-fixed ($\alpha = 0$) with the tangential boundary conditions in (3), namely, $T_{1n}(0) = V_n(0) = 0$, and the other edge free ($\alpha = \alpha_1$) with the tangential conditions in (2), namely, $T_{1n}(\alpha_1) = S_n(\alpha_1) = 0$:

$$\frac{Eh}{p_0R^2} u(\alpha,\beta) = \frac{1}{2\mu_n^2} [\bar{S}_n(0)\Phi_4(\alpha) - U_n^*(0)\Phi_2(\alpha) + \Phi_4(\alpha-\xi_2) - \Phi_4(\alpha-\xi_1)] \cos n\beta,$$

$$\frac{Eh}{p_0R^2} w(\alpha,\beta) = \frac{n^2}{4\mu_n^3} \bar{w}_n(\alpha) \cos n\beta,$$

$$\frac{1}{p_0R^2} G_2(\alpha,\beta) = \frac{\mu_n}{n^2(n^2-1)} \bar{G}_{2n}(\alpha) \cos n\beta, \tag{14}$$

$$\bar{w}_n(\alpha) = \bar{G}_{2n}(\alpha) = \bar{S}_n(0)[\Phi_1(\alpha) - \Phi_3(\alpha)] - U_n^*(0)[\Phi_1(\alpha) + \Phi_3(\alpha)] + \Phi_1(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) - \Phi_3(\alpha-\xi_2) + \Phi_3(\alpha-\xi_1),$$

TABLE 4. Dimensionless Values of the Displacements, Forces, and Moment under the Effect of a Piecewise-Continuous Longitudinal Load on an Envelope with the Parameters $R/h = 100$, $l/R = 4$

Factors sought	Longitudinal coordinate α				
	0	1.75	2	2.25	4
$\overline{u}$	0.616	0.624	0.628	0.630	1.14
$\overline{w}$	0	0.312	0.338	0.380	0.68
$\overline{T}_1$	0	0.025	0.023	−0.053	0
$\overline{S}$	0	−0.150	−0.137	−0.120	0
$\overline{G}_2$	0	0.459	0.497	0.558	1

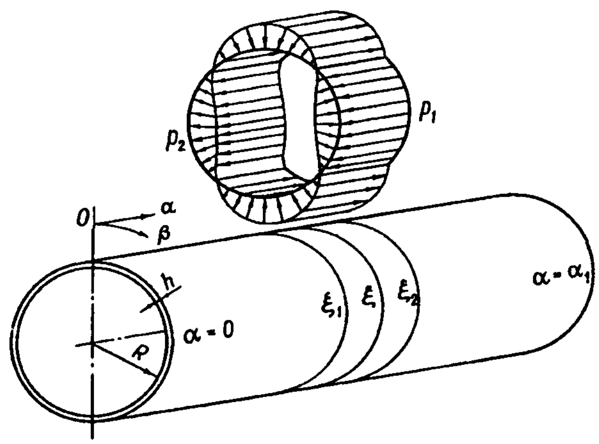

Fig. 1. An envelope under the effect of a piecewise-cosinusoidal longitudinal or normal load.

$$\frac{1}{p_0 R} T_1(\alpha, \beta) = \frac{1}{2\mu_n} \Big\{ \overline{S}_n(0) [\Phi_1(\alpha) + \Phi_3(\alpha)] - U_n^*(0) [\Phi_3(\alpha) - \Phi_1(\alpha)] +$$

$$+ \Phi_1(\alpha - \xi_2) - \Phi_1(\alpha - \xi_1) + \Phi_3(\alpha - \xi_2) - \Phi_3(\alpha - \xi_1) \Big\} \cos n\beta ,$$

$$\frac{1}{p_0 R} S(\alpha, \beta) = \frac{1}{n} [- \overline{S}_n(0) \Phi_2(\alpha) - \overline{U}_n^*(0) \Phi_4(\alpha) + \Phi_2(\alpha - \xi_1) - \Phi_2(\alpha - \xi_2)] \sin n\beta ,$$

$$C(\alpha_1) \overline{S}_n(0) = - A(\alpha_1) \Phi_4(\alpha_1) + B(\alpha_1) [\Phi_3(\alpha_1) - \Phi_1(\alpha_1)] ,$$

$$C(\alpha_1) U_n^*(0) = B(\alpha_1) [\Phi_1(\alpha_1) + \Phi_3(\alpha_1)] + A(\alpha_1) \Phi_2(\alpha_1) ,$$

$$A(\alpha_1) = \Phi_1(\alpha_1 - \xi_2) - \Phi_1(\alpha_1 - \xi_1) + \Phi_3(\alpha_1 - \xi_2) - \Phi_3(\alpha_1 - \xi_1) ,$$

$$B(\alpha_1) = \Phi_2(\alpha_1 - \xi_1) - \Phi_2(\alpha_1 - \xi_2) ,$$

$$C(\alpha_1) = \Phi_4(\alpha_1) [\Phi_1(\alpha_1) + \Phi_3(\alpha_1)] + \Phi_2(\alpha_1) [\Phi_3(\alpha_1) - \Phi_1(\alpha_1)] .$$

Let us note that solution (14) is written for the numbers of harmonics $n \geq 2$. But, in accordance with the problem posed, the case $n = 2$, when the stressed-and-strained state varies along the generatrix most slowly, is of primary interest to us. Building the solution (14) not only makes it possible to obtain a qualitative picture of the stressed state but also makes it possible to find rather exact values of the factors sought: the displacements and the stresses [7]. Results of calculations on the basis of (14) are presented in Table 4. The envelope is loaded by a force p_0 cos

TABLE 5. Dimensionless Values of the Displacements, Force, and Bending Moment under the Effect of a Piecewise-Continuous Longitudinal Load on Envelopes of Different Length with the Parameters $l/R = 4, 8, 16$

Factors sought	α_1	Longitudinal coordinate α				
		0	ξ_1	ξ	ξ_2	α_1
$\bar{u}$	4	4.830	4.843	4.846	4.843	4.83
	8	0.602	0.626	0.629	0.626	0.602
	16	0.061	0.113	0.114	0.113	0.061
$\bar{w}$	4	−2.620	−0.320	0	0.320	2.620
	8	−0.672	−0.050	0	0.050	0.672
	16	−0.163	−0.007	0	0.007	0.163
$\bar{T}_1$	4	0	0.100	0	−0.100	0
	8	0	0.122	0	−0.122	0
	16	0	0.130	0	−0.130	0
$\bar{G}_2$	4	1	0.122	0	−0.122	1
	8	1	0.074	0	−0.074	1
	16	1	0.043	0	−0.043	1

2β that is piecewise-continuous in the longitudinal direction on a section of length $2\alpha_0 R = 0.5R$, located in the middle of the envelope (Fig. 1).

On the basis of the solution presented in Tables 2 and 3, other variants of the boundary conditions, including envelopes with the both edges free, with parameters $R/h = 100$, $\alpha_1 = l/R = 4, 8, 16$, are considered. Numerical information in dimensionless form for envelopes with free edges is presented for character points of them in Table 5. Here, as well as in Table 4 and later on, it is agreed that $\beta = 0$.

3. Let us touch upon a case that is very important for understanding the problem posed, when a radial step load affects envelopes of finite length and a cosinusoidal load acts around the circumference.

In the same manner as in the preceding case, we apply the semi-zero-moment theory of envelopes, for which the resolving equation is written in the following form:

$$\mathscr{L}\Phi(\alpha, \beta) = (R^2/Eh)\, p(\alpha, \beta)\,, \tag{15}$$

where $\mathscr{L}(\ )$ is the operator of semi-zero-moment theory that was used before (12).

The solution on the basis of the semi-zero-moment theory (15) built by the method of initial parameters is presented in the form of Table 6, in which the notation introduced before is kept. The radial load load acting on the section of the generatrix $[\xi_1, \xi_2]$ may vary arbitrarily over this section or else may remain constant. For such variants the solution is written in the next to last and last columns of the mentioned table, respectively.

For the most important sought factors of the stressed-and-strained state of an envelope having one edge free ($\alpha = 0$) and the other edge rigidly pinched ($\alpha = \alpha_1$), analytical expressions are given below:

$$\frac{Eh}{p_0 R^2} u(\alpha, \beta) = \frac{n^2}{4\mu_n^3} \Big\{ -\bar{V}_n^*(0)\, [\Phi_3(\alpha) - \Phi_1(\alpha)] + \bar{U}_n^*(0)\, \Phi_2'(\alpha) -$$

$$- \Phi_1(\alpha - \xi_2) + \Phi_1(\alpha_1 - \xi_1) + \Phi_3(\alpha - \xi_2) - \Phi_3(\alpha - \xi_1) \Big\} \cos n\beta\,,$$

$$\frac{Eh}{p_0 R^2} w(\alpha, \beta) = \frac{n^4}{8\mu_n^4} \bar{w}_n(\alpha) \cos n\beta\,;$$

TABLE 6. Solution for the Case of the Effect of a Normal Load $p_0 \cos n\beta$ That Is Arbitrarily Varying and Piecewise-Continuous on the Section $[\xi_1, \xi_2]$

Amplitude values of the forces and displacements	$T_{1n}(0)$	$S_n(0)$	$V_n^*(0)$	$U_n^*(0)$	p_0R	
$T_{1n}(\alpha)$	$K_{TT}(\alpha)$	$K_{TS}(\alpha)$	$K_{TV}(\alpha)$	$K_{TU}(\alpha)$	$\frac{n^2}{2\mu_n}\int_{\xi_1}^{\xi_2}[\Phi_1(\alpha-\xi)+\Phi_3(\alpha-\xi)]\theta(\xi)d\xi$	$-\frac{n^2}{2\mu_n^2}[\Phi_4(\alpha-\xi_2)-\Phi_4(\alpha-\xi_1)]$
$S_n(\alpha)$	$K_{ST}(\alpha)$	$K_{SS}(\alpha)$	$K_{SV}(\alpha)$	$K_{SU}(\alpha)$	$-n\int_{\xi_1}^{\xi_2}\Phi_2(\alpha-\xi)\theta(\xi)d\xi$	$\frac{n}{2\mu_n}[\Phi_1(\alpha-\xi_2)-\Phi_1(\alpha-\xi_1)+\Phi_3(\alpha-\xi_2)-\Phi_3(\alpha-\xi_1)]$
$V_n^*(\alpha)$	$K_{VT}(\alpha)$	$K_{VS}(\alpha)$	$K_{VV}(\alpha)$	$K_{VU}(\alpha)$	$-\frac{n^3}{4\mu_n^3}\int_{\xi_1}^{\xi_2}[\Phi_3(\alpha-\xi)-\Phi_1(\alpha-\xi)]\theta(\xi)d\xi$	$\frac{n^2}{4\mu_n^4}[\Phi_2(\alpha-\xi_2)-\Phi_2(\alpha-\xi_1)$
$U_n^*(\alpha)$	$K_{UT}(\alpha)$	$K_{US}(\alpha)$	$K_{UV}(\alpha)$	$K_{UU}(\alpha)$	$\frac{n^2}{2\mu_n^2}\int_{\xi_1}^{\xi_2}\Phi_4(\alpha-\xi)\theta(\xi)d\xi$	$-\frac{n^2}{4\mu_n^3}[\Phi_1(\alpha-\xi_2)-\Phi_1(\alpha-\xi_1)-\Phi_3(\alpha-\xi_2)+\Phi_3(\alpha-\xi_1)]$

$$\frac{1}{p_0R^2}G_2(\alpha,\beta)=\frac{1}{2(n^2-1)^2}\overline{G}_{2n}(\alpha)\cos n\beta,$$

$$\overline{w}_n(\alpha)=\overline{G}_{2n}(\alpha)=-2\overline{V}_n^*(0)\Phi_2(\alpha)+$$

$$+\overline{U}_n^*(0)[\Phi_1(\alpha)+\Phi_3(\alpha)]+2[\Phi_2(\alpha-\xi_2)-\Phi_2(\alpha-\xi_1)],$$

$$\frac{1}{p_0R}T_1(\alpha,\beta)=\frac{n^2}{4\mu_n^2}\overline{T}_{1n}(\alpha)\cos n\beta,$$

$$\overline{T}_{1n}(\alpha)=2\overline{V}_n^*(0)\Phi_4(\alpha)+\overline{U}_n^*(0)[\Phi_3(\alpha)-\Phi_1(\alpha)]-2[\Phi_4(\alpha-\xi_2)-\Phi_4(\alpha-\xi_1)],$$

$$C(\alpha_1)\overline{V}_n^*(0)=A(\alpha_1)\Phi_2(\alpha_1)+B(\alpha_1)[\Phi_1(\alpha_1)+\Phi_3(\alpha_1)], \tag{16}$$

$$C(\alpha_1)\overline{U}_n^*(0)=2B(\alpha_1)\Phi_2(\alpha_1)+A(\alpha_1)[\Phi_3(\alpha_1)-\Phi_1(\alpha_1)],$$

$$A(\alpha_1)=\Phi_2(\alpha_1-\xi_2)-\Phi_2(\alpha-\xi_1),$$

$$B(\alpha_1)=\frac{1}{2}[\Phi_1(\alpha_1-\xi_2)-\Phi_1(\alpha_1-\xi_1)-\Phi_3(\alpha_1-\xi_2)+\Phi_3(\alpha_1-\xi_1)],$$

$$C(\alpha_1)=\Phi_2^2(\alpha_1)-\frac{1}{2}[\Phi_3^2(\alpha_1)-\Phi_1^2(\alpha_1)].$$

These formulas are valid for $n \geq 2$.

TABLE 7. Dimensionless Values of the Displacements, Normal Force, and Bending Moment under the Effect of a Normal Load $p_0 \cos 2\beta$ That Is Piecewise-Continuous on the Section $[\xi_1, \xi_2]$ Acting on Envelopes with Different Boundary Conditions

Boundary conditions		Factors sought	Longitudinal coordinate α				
edge $\alpha = 0$	edge $\alpha = 8$		0	3.75	4	4.25	8
Free	Free	$\bar{w}$	0.133		0.135		0.133
		$\bar{T}_1$	0	−0.040	−0.042	−0.040	0
		$\bar{G}_2$	0.985	1.00	1.00	1.00	0.985
Rigid	Rigid	$\bar{w}$	0		0.004		0
		$\bar{T}_1$	0.030		−0.050		0.030
Free	Rigid	$\bar{u}$	−0.026		−0.029		0
		$\bar{w}$	0.052	0.029	0.023	0.021	0
		$\bar{T}_1$	0		−0.015		0.053

Besides the case noted before, a solution is obtained for envelopes with the following boundary conditions: both edges free; a free edge plus one edge hinge-fixed; both edges fixed. Some numerical results for envelopes with the boundary conditions mentioned before and having the parameters $R/h = 100$, $l/R = 8$, $\xi_1 = 3.75$, $\xi_2 = 4.25$, $\alpha_0 = 0.25$ are given in Table 7. The force $p_0 \cos 2\beta$ affected the envelopes.

4. Now let us touch upon the case where the temperature field $t^*(\alpha, \beta)$, constant over the thickness but arbitrarily varying over its surface, appears in the envelope. If the semi-zero-moment theory is used, as was done for the effect pf force loads, the resolving equation can be presented in the following form [6]:

$$\mathscr{L}\Phi^*(\alpha, \beta) = \frac{1}{1-\nu} \alpha_t R t^*(\alpha, \beta), \tag{17}$$

where $\mathscr{L}(\ )$ is the operator of the semi-zero-moment theory of envelopes (12).

Displacements, force, and bending moments are expressed in terms of the resolving fucntion by differential relations.

The solution for a temperature field localized in the longitudinal direction that is of interest to us can be rather easily written on the basis of using the method of initial parameters. Complications related to building particular solutions in using the method of initial parameters in thermoelastic problems are surmountable if the mathematical thermal-force analogy between the effects in the envelope caused by the action of a force load and a temperature field of the same distribution law over the surface of the envelope is used [6].

Use of the semi-zero-moment theory of envelopes makes it possible to give the simplest form of the relationships of the thermal-force analogy:

$$u^*(\alpha, \beta) = -f^* \frac{\partial u(\alpha, \beta)}{\partial \alpha}, \quad w^*(\alpha, \beta) = -f^* \frac{\partial w(\alpha, \beta)}{\partial \alpha},$$

$$T_1^*(\alpha, \beta) = -f^* \frac{\partial T_1(\alpha, \beta)}{\partial \alpha} - \frac{Eh}{1-\nu} \alpha_t t^*(\alpha, \beta), \tag{18}$$

$$S^*(\alpha, \beta) = -f^* \frac{\partial S(\alpha, \beta)}{\partial \alpha}, \quad G_2^*(\alpha, \beta) = -f^* \frac{\partial G_2(\alpha, \beta)}{\partial \alpha}, \quad f^* = \frac{\alpha_t E t_0^*}{(1-\nu) p_0} \frac{h}{R}.$$

TABLE 8. Solution for the Case of an Arbitrary and Piecewise-Continuous Distribution of the Temperature Field on the Section $[\xi_1, \xi_2]$ of the Generatrix and a Temperature Field $t_0^* \cos n\beta$ That Varies Cosinusoidally around the Circumference

Amplitude values of the forces and displacements	$T_{1n}(0)$	$S_n(0)$	$V_n^*(0)$	$U_n^*(0)$	$\frac{\alpha_t E h t_0^*}{1-\nu}$	
$T_{1n}(\alpha)$	$K_{TT}(\alpha)$	$K_{TS}(\alpha)$	$K_{TV}(\alpha)$	$K_{TU}(\alpha)$	$\frac{\mu_n}{n}\int_{\xi_1}^{\xi_2}[\Phi_3(\alpha-\xi) - \Phi_1(\alpha-\xi)]\theta(\xi)d\xi$	$-\Phi_2(\alpha-\xi_2)+\Phi_2(\alpha-\xi_1)$ $(\alpha>\xi_2)$ $\Phi_2(\alpha-\xi_1)-1$ $(\xi_1\le\alpha\le\xi_2)$ 0 $(\alpha<\xi_1)$
$S_n(\alpha)$	$K_{ST}(\alpha)$	$K_{SS}(\alpha)$	$K_{SV}(\alpha)$	$K_{ST}(\alpha)$	$\frac{2\mu_n^2}{n^2}\int_{\xi_1}^{\xi_1}\Phi_4(\alpha-\xi)\theta(\xi)d\xi$	$-\frac{\mu_n}{n}[\Phi_3(\alpha-\xi_1)-\Phi_1(\alpha-\xi_1) - \Phi_3(\alpha-\xi_2)+\Phi_1(\alpha-\xi_2)]$
$V_n^*(\alpha)$	$K_{VT}(\alpha)$	$K_{VS}(\alpha)$	$K_{VV}(\alpha)$	$K_{VU}(\alpha)$	$\frac{1}{2\mu_n}\int_{\xi_1}^{\xi_2}[\Phi_1(\alpha-\xi) + \Phi_3(\alpha-\xi)]\theta(\xi)d\xi$	$-\frac{n}{2\mu_n^2}[\Phi_4(\alpha-\xi_2)-\Phi_4(\alpha-\xi_1)]$
$U_n^*(\alpha)$	$K_{UT}(\alpha)$	$K_{US}(\alpha)$	$K_{UV}(\alpha)$	$K_{UU}(\alpha)$	$\frac{1}{n}\int_{\xi_1}^{\xi_2}\Phi_2(\alpha-\xi)\theta(\xi)d\xi$	$-\frac{1}{2\mu_n}[\Phi_1(\alpha-\xi_2)+\Phi_3(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1)-\Phi_3(\alpha-\xi_1)]$

From (18) is evident that in the case of a temperature field $t^*(\alpha, \beta)$ affecting an envelope the essence of the thermal-force analogy is as follows: the temperature displacements, forces, bending moments, and other factors u^*, v^*, w^*, T_1^*, G_2^*, ... are expressed in terms of the corresponding factors u, v, ..., G_2 appearing in the envelope under the effect of the longitudinal load (5). Using the relationships of the thermal-force analogy (18) in the solution for the case of the action of a longitudinal load written in the form of Table 3, one can easily build a particular solution under a temperature effect and then write down a complete solution of the problem. The latter is presented in the form of Table 8 and was used in solving boundary-value problems for the difference equation (17) with boundary conditions (2)-(4). Here, we restrict our consideration to presentation of an analytical solution of the boundary-value problem for an envelope with both edges free, where $T_{1n}(0) = S_n(0) = 0$, $T_{1n}(\alpha_1) = S_n(\alpha_1) = 0$:

$$\frac{1-\nu}{\alpha_t R t_0^*}u(\alpha,\beta) = -\frac{1}{2\mu_n}\left\{\frac{1}{2}\overline{V}_n^*(0)\,[\Phi_3(\alpha)-\Phi_1(\alpha)] + \overline{U}_n^*(0)\,\Phi_2(\alpha) + \right.$$

$$\left. + \Phi_1(\alpha-\xi_2) + \Phi_3(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) + \Phi_3(\alpha-\xi_1)\right\}\cos n\beta\,,$$

$$\frac{1-\nu}{\alpha_t R t_0^*}w(\alpha,\beta) = -\frac{n^2}{4\mu_n^2}\overline{w}_n(\alpha)\cos n\beta\,,$$

$$\left(\frac{\alpha_t E t_0^*}{1-\nu}\frac{h^2}{6}\right)^{-1} G_2(\alpha,\beta) = -\frac{3}{2\sqrt{3(1-\nu^2)}}\overline{G}_{2n}(\alpha)\cos n\beta\,,$$

$$\overline{w}_n(\alpha) = \overline{G}_{2n}(\alpha) = \overline{V}_n^*(0)\,\Phi_2(\alpha) + \overline{U}_n^*(0)\,[\Phi_1(\alpha)+\Phi_3(\alpha)] +$$

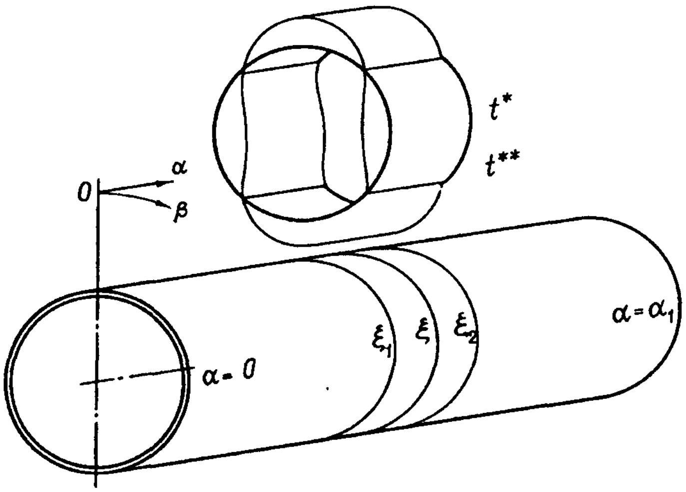

Fig. 2. An envelope under the effect of a piecewise-cosinusoidal temperature field.

$$+ 2\,[\Phi_4(\alpha-\xi_2) - \Phi_4(\alpha-\xi_1)]\,,$$

$$\frac{1-\nu}{\alpha_t E h t_0^*} T_1(\alpha,\beta) = \frac{1}{2}\left\{ \overline{V}_n^*(0)\,\Phi_4(\alpha) - \overline{U}_n^*(0)\,[\Phi_3(\alpha) - \Phi_1(\alpha)] + 2\varphi(\alpha)\right\} \cos n\beta\,,$$

$$\varphi(\alpha) = 0 \quad (\alpha < \xi_1)\,,\quad \varphi(\alpha) = \Phi_2(\alpha-\xi_1) - 1 \quad (\xi_1 \le \alpha \le \xi_2)\,,$$

$$\varphi(\alpha) = \Phi_2(\alpha-\xi_1) - \Phi_2(\alpha-\xi_2) \quad (\alpha > \xi_2)\,, \tag{19}$$

$$\frac{1-\nu}{\alpha_t E h t_0^*} S(\alpha,\beta) = -\frac{\mu_n}{n}\left\{\frac{1}{2}\overline{V}_n^*(0)\,[\Phi_1(\alpha) + \Phi_3(\alpha)] + \right.$$

$$\left. + \overline{U}_n^*(0)\,\Phi_4(\alpha) + \Phi_3(\alpha-\xi_1) - \Phi_1(\alpha-\xi_1) - \Phi_3(\alpha-\xi_2) + \Phi_1(\alpha-\xi_2)\right\} \sin n\beta\,,$$

$$C(\alpha_1)\,\overline{V}_n^*(0) = 2B(\alpha_1)\,\Phi_4(\alpha_1) - A(\alpha_1)\,[\Phi_3(\alpha_1) - \Phi_1(\alpha_1)]\,,$$

$$C(\alpha_1)\,\overline{U}_n^*(0) = -A(\alpha_1)\,\Phi_4(\alpha_1) + B(\alpha_1)\,[\Phi_1(\alpha_1) + \Phi_3(\alpha_1)]\,,$$

$$A(\alpha_1) = \Phi_3(\alpha_1-\xi_1) - \Phi_1(\alpha_1-\xi_1) - \Phi_3(\alpha_1-\xi_2) + \Phi_1(\alpha_1-\xi_2)\,,$$

$$B(\alpha_1) = \Phi_2(\alpha_1-\xi_1) - \Phi_2(\alpha_1-\xi_2)\,,$$

$$C(\alpha_1) = \Phi_4^2(\alpha_1) + \frac{1}{2}\,[\Phi_3^2(\alpha_1) - \Phi_1^2(\alpha_1)]\,,\quad n \ge 2\,.$$

Analytical expressions (19) became the basis for obtaining numerical information on the stressed-and-strained state of an envelope with the parameters $R/h = 100$, $\alpha_1 = l/R = 8$. The temperature field $T_0^* \cos 2\beta$ affects the section of the envelope's surface $[\xi_1 = 3.75,\ \xi_2 = 4.25]$ (Fig. 2). Results of calculations of the sought factors at the most important points of the envelope are presented in Table 9. In this table information for boundary-value problems with other boundary conditions is given: with both edges rigidly pinched, and also with one edge free and the other rigidly pinched.

TABLE 9. Dimensionless Values of the Displacements, Force, and Moment under the Effect of a Temperature Field $t_0^* \cos 2\beta$ That Is Piecewise-Continuous on a Section of the Generatrix Acting on an Envelope with the Parameter $l/R = 8$

Boundary conditions		Factors sought	Longitudinal coordinate α				
edge $\alpha = 0$	edge $\alpha = 8$		0	3.75	4	4.25	8
Free	Free	$\bar{u}$	0.0749	0.0757	0	−0.757	−0.0749
		$\bar{w}$	−0.0392	0.0376	0.0405	0.0376	−0.0392
		$\bar{T}_1$	0	−0.0035	−0.0040	−0.0035	0
		$\bar{G}_2$	0.968	−0.928	−1.000	−0.928	0.968
Rigid	Rigid	$\bar{u}$	0		0		0
		$\bar{w}$	0		0.0346		0
		$\bar{T}_1$	0.124		−0.118		0.124
Free	Rigid	$\bar{w}$	−0.110	−0.016	−0.012	−0.012	0

5. Let us consider the case where the temperature field is characterized by a difference over the thickness of the envelope, i.e., where the temperature field $t^{**}(\alpha, \beta)$ occurs. The resolving equation of the semi-zero-moment theory of envelopes can be written in the following form [6]:

$$\mathscr{L}\Phi^{**}(\alpha, \beta) = -\frac{1}{6(1-\nu)} \alpha_t h t^{**}(\alpha, \beta), \tag{20}$$

where $\mathscr{L}(\ )$ is the operator of the semi-zero-moment theory (12).

In this case we shall also use relationships of the mathematical thermal-force analogy between the effects from the action of a force load and a temperature field $t^{**}(\alpha, \beta)$. Here, the essence of the thermal-force analogy is as follows: all factors of the thermoelastic problem can be found if the corresponding factor with the radial load present is affected by the differential operator $(\partial^2/\partial\beta^2 + 1)$ and then the expression obtained is multiplied by the coefficient f^{**} taken with a "−" sign:

$$u^{**} = -(\partial^2/\partial\beta^2 + 1)\, u(\alpha, \beta) f^{**}; \quad v^{**} = -(\partial^2/\partial\beta^2 + 1)\, v(\alpha, \beta) f^{**},$$

$$w^{**} = -(\partial^2/\partial\beta^2 + 1)\, w(\alpha, \beta) f^{**}; \quad T_1^{**} = -(\partial^2/\partial\beta^2 + 1)\, T_1(\alpha, \beta) f^{**},$$

$$G_2^{**} = -(\partial^2/\partial\beta^2 + 1)\, G_2(\alpha, \beta) f^{**} - \frac{\alpha_t E t^{**}(\alpha, \beta)}{1-\nu} \frac{h^2}{6}, \tag{21}$$

$$S^{**} = -(\partial^2/\partial\beta^2 + 1)\, S(\alpha, \beta) f^{**}, \quad f^{**} = \frac{\alpha_t E t_0^{**}}{6(1-\nu) p_0} \left(\frac{h}{R}\right)^2.$$

It is reasonable to build the solution on the basis of the method of initial parameters. Here, the general solution written down in Table 10 coincides with the solution in the left-hand part of Table 6. As concerns the particular solution in Table 10, it is obtained by applying relationships (21) to the particular solution in the right-hand part of Table 6. Boundary-value problems for the differential equation (20) with boundary conditions (2)-(4) are considered on the basis of the solution written down in Table 10. As an example we shall give analytical expressions only for an envelope having one edge free ($\alpha = 0$) and the other rigidly pinched ($\alpha = \alpha_1$):

TABLE 10. Solution for the Case of a Temperature Field $t_0^{**} \cos n\beta$ That Varies Arbitrarily and Is Piecewise-Continuous on the Section $[\xi_1, \xi_2]$ of the Generatrix

Amplitude values of the forces and displacements	$T_{1n}(0)$	$S_n(0)$	$V_n^*(0)$	$U_n^*(0)$	$\frac{\alpha_t E t_0^{**}}{1-\nu} \frac{h^2}{6} n(n^2-1)R^{-1}$	
$T_{1n}(\alpha)$	$K_{TT}(\alpha)$	$K_{TS}(\alpha)$	$K_{TV}(\alpha)$	$K_{TU}(\alpha)$	$\frac{n}{2\mu_n} \int_{\xi_1}^{\xi_2} [\Phi_1(\alpha-\xi) + \Phi_3(\alpha-\xi)]\theta(\xi)d\xi$	$-\frac{n}{2\mu_n^2}[\Phi_4(\alpha-\xi_2) - \Phi_4(\alpha-\xi_1)]$
$S_n(\alpha)$	$K_{ST}(\alpha)$	$K_{SS}(\alpha)$	$K_{SV}(\alpha)$	$K_{SU}(\alpha)$	$-\int_{\xi_1}^{\xi_2} \Phi_2(\alpha-\xi)\theta(\xi)d\xi$	$\frac{1}{2\mu_n}[\Phi_1(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) + \Phi_3(\alpha-\xi_2) - \Phi_3(\alpha-\xi_1)]$
$V_n^*(\alpha)$	$K_{VT}(\alpha)$	$K_{VS}(\alpha)$	$K_{VV}(\alpha)$	$K_{VU}(\alpha)$	$-\frac{n^2}{4\mu_n^3} \int_{\xi_1}^{\xi_2} [\Phi_3(\alpha-\xi) - \Phi_1(\alpha-\xi)]\theta(\xi)d\xi$	$\frac{n^2}{4\mu_n^4}[\Phi_2(\alpha-\xi_2) - \Phi_2(\alpha-\xi_1)]$
$U_n^*(\alpha)$	$K_{UT}(\alpha)$	$K_{US}(\alpha)$	$K_{UV}(\alpha)$	$K_{UU}(\alpha)$	$\frac{n}{2\mu_n^2} \int_{\xi_1}^{\xi_2} \Phi_4(\alpha-\xi)\theta(\xi)d\xi$	$-\frac{n}{4\mu_n^3}[\Phi_1(\alpha-\xi_2) - \Phi_1(\alpha-\xi_1) - \Phi_3(\alpha-\xi_2) + \Phi_3(\alpha-\xi_1)]$

$$\frac{1}{\alpha_t R t_0^{**}} w(\alpha, \beta) = \frac{1+\nu}{n^2-1} \frac{R}{h} \overline{w}_n(\alpha) \cos n\beta,$$

$$\frac{1-\nu}{\alpha_t E h t_0^{**}} T_1(\alpha, \beta) = \frac{1}{24\mu_n^2} n^2 (n^2-1) \overline{T}_{1n}(\alpha) \cos n\beta, \tag{22}$$

$$\left(\frac{\alpha_t E t_0^{**}}{1-\nu} \frac{h^2}{6}\right)^{-1} G_2(\alpha, \beta) = \left[\frac{1}{2} \overline{G}_{2n}(\alpha) - \theta(\alpha)\right] \cos n\beta.$$

The initial parameters $\overline{V}_n^*(0)$, $\overline{U}_n^*(0)$ and the functions $\overline{w}_n(\alpha)$, $\overline{T}_{1n}(\alpha)$, $\overline{G}_{2n}(\alpha)$ in the expressions (22) coincide with the ones given in (16) for the case of a normal load affecting the envelope; $\theta(\alpha)$ is the dimensionless piecewise-continuous, in a particular case step, function of the temperature distribution on the section $[\xi_1, \xi_2]$ of the generatrix.

Numerical information for envelopes with the parameters $R/h = 100$, $l/R = 8$ is presented in Table 11 for a temperature field $t_0^{**} \cos 2\beta$ affecting the section $\xi_1 = 3.75$, $\xi_2 = 4.25$ of the surface, and therefore the length of the heated zone amounts to $2\alpha_0 R = 0.5R$ (Fig. 2). The following combinations of boundary conditions are considered: both edges are free, both edges are rigidly pinched, and one edge is free and the other is rigidly pinched.

6. Let us note that the numerical information in all the tables is given in a dimensionless form that is convenient for perception of the most important displacements, forces, and bending moments at characteristic points. In some cases the maximum values of some factors were taken to be unity, which is why comparison of values of the sought factors for different boundary-value problems is not legitimate. In further considerations in all cases the force load and the temperature field over the section of their action $[\xi_1, \xi_2]$ were considered constant, which is most unfavorable for making use of the St. Venant principle.

TABLE 11. Dimensionless Values of the Displacements and the Longitudinal Force in Envelopes with Different Boundary Conditions under the Effect of a Temperature Field t_0^{**} cos 2β Having a Difference over the Thickness, $l/R = 8$

Boundary conditions		Factors sought	Longitudinal coordinate α				
edge $\alpha = 0$	edge $\alpha = 8$		0	3.75	4	4.25	8
Free	Free	$\overline{w}$	0.985		1		0.985
		$\overline{T}_1$	0	−0.952	−1	−0.952	0
Rigid	Rigid	$\overline{u}$	0		0		0
		$\overline{w}$	0		1		0
		$\overline{T}_1$	0.600		−1		0.600
Free	Rigid	$\overline{u}$	−1	−1.115			0
		$\overline{w}$	1	0.558	0.442	0.404	0
		$\overline{T}_1$	0		−0.283		1

Let us consider the numerical information given in the form of the tables. Numerical information for the effect of the longitudinal load p_0 cos 2β (Table 4) is presented for all the most interesting displacements, forces, and bending moment in the middle of the loaded region ($\alpha = \xi$), on its boundaries ($\alpha = \xi_1$, $\alpha = \xi_2$), and on the edges of an envelope whose length equals four radii ($l = 4R$). All the factors are given for the zero generatrix $\beta = 0$, where they take maximum values. It is seen that the boundary conditions are satisfied. The strain of the envelope occurs in such a way that the normal displacement $\overline{w}$ is positive along the entire zero generatrix, which corresponds to reduction of the radius. Here, it increases from zero on the edge $\alpha = 0$, where a rigid diaphragm occurs, to the value $\overline{w} = 0.68$ on the free edge, which is more than twice the normal displacement in the zone of application of the load. The deplanation of the cross section on the free edge $\overline{u}(\alpha_1)$ is also almost twice the deplanation in the region of the load's action, e.g. $\overline{u}(\xi_1)$, $\overline{u}(\xi)$, $\overline{u}(\xi_2)$. The longitudinal and shearing forces $\overline{T}_1$, $\overline{S}$ have substantial values beyond the zone of the load's action, and the circular bending moment $\overline{G}_2$ and the normal displacement $\overline{w}$ grow substantially as they move out of the loaded region toward the free edge, reaching a maximum on it. Values of the longitudinal displacement $\overline{u}$, which characterizes the deplanation of the cross sections of the envelope, the normal displacement $\overline{w}$, the longitudinal force $\overline{T}_1$, and the circular moment $\overline{G}_2$ for envelopes with free edges having different elongations $\alpha_1 = l/R = 4, 8, 16$ are presented in Table 5. It is of interest to note that even for envelopes having substantial elongation, as in Table 5, the deplanation along the entire generatrix, from one edge to the other, practically retains its value. For an envelope with elongation $l/R = 16$ the value of $\overline{u}$ on the edges is approximately half that of the loaded region, i.e., some damping is seen as it moves toward the edges. But, in accordance with the St. Venant principle, which was formulated, as is known, for continuous bodies, the disturbance caused by the load must diminish substantially at a distance $x > 2R$ from the point of the self-balanced load's application, which is clearly not seen in the considered cases. The normal displacement $\overline{w}$, which characterizes here only deviations from the circular form, increases as it approaches the edges, being antisymmetric relative to the middle of the envelope $\alpha = \xi$. The circular bending moment $\overline{G}_2$ grows substantially as it approaches the edges of the envelope, exceeding manifold the values in the loaded region.

The numerical information for the case of a radial load p_0 cos $2\beta\alpha$ acting over the section of the surface $[\xi_1, \xi_2]$ given in Table 7 for envelopes with various boundary conditions does not contradict the vast information presented previously for a longitudinal load. Thus, the most characteristic quantities $\overline{w}$, $\overline{G}_2$ for an envelope with free edges practically do not change as they move out of the loaded zone toward the edges: the reduction amounts to approximately 2%. For envelopes with rigidly pinched edges the longitudinal force $\overline{T}_1$ on the edges amounts to 60% of the force in the middle of the loaded zone. The normal displacement $\overline{w}$ on the free edge that is characteristic for envelopes with one edge free and the other pinched is more than twice the shift in the middle of the loaded zone.

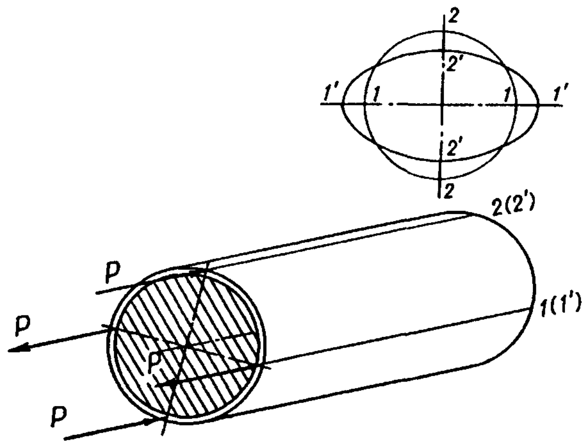

Fig. 3. Layout for loading an envelope with a self-balanced system of longitudinal concentrated forces in an experimental investigation: 1, 2, 1', 2') location of diametrically opposite points of the free edge of the envelope before and after application of the load, respectively.

If the temperature field $t_0^* \cos 2\beta$ appears in the envelope on the section $[\xi_1, \xi_2]$, numerical information for different boundary-value problems is presented in Table 9 at the characteristic points $\alpha = 0$, ξ_1, ξ, ξ_2, α_1. In an envelope with both edges free the deplanation $\overline{u}$ grows from zero in its middle up to some values on the border of the "hot" region that differ insignificantly from the deplanation on the edges of the envelope. The radial displacement $\overline{w}$ changes its sign as it moves out of the middle of the "hot" zone toward the edges, on which it takes a value that is only several percent less than that in the middle of the "hot" zone. The same picture can be seen in considering the circular bending moment. In an envelope with rigidly fixed edges the longitudinal force $\overline{T}_1$ in the embedment is greater than that in the middle of the envelope. In an envelope with one edge free and the other fixed the normal shift $\overline{w}$ on the free edge is an order greater than in the "hot" zone.

The appearance of the temperature field $t_0^{**} \cos 2\beta$ on the section $[\xi_1, \xi_2]$ of the envelope causes a substantial disturbance at a large distance from the "hot" zone. Thus, in an envelope with free edges (Table 11) the normal shift $\overline{w}$ on the edges differs from $\overline{w}$ in the middle of the "hot" zone by only several percent, and in an envelope with one edge free and the another pinched $\overline{w}$ on the free edge is almost twice that in the middle of the envelope. In an envelope with rigidly pinched edges the longitudinal force $\overline{T}_1$ in the middle and on the edges has values of the same order.

In all cases of loading and heating of envelopes with marginal boundary conditions – free edges, rigid edges, hinge-fixed edges – even at great elongation ($l/R = 8$) the edges of the envelopes are exposed to disturbances under loading and heating in the middle section of the envelopes.

Consideration of the behavior of envelopes under the action of self-balanced systems of local loads, concentrated forces, and localized temperature fields is of some interest. In solving such boundary-value problems with the aim of investigation of the behavior of the most slowly damping part of the stressed state the solutions built here can be used if they are multiplied by the corresponding Fourier coefficients of expansion in terms of the circumferential coordinate. Now let us present results of an experimental investigation of envelopes under the effect of a system of longitudinal concentrated forces. The layout of the loading of an envelope with one edge free and the another hinge-fixed is presented in Fig. 3. The load was applied to an edge containing a diaphragm rigid in its plane. Under the action of a load $P = 9.8$ N the free edge of the envelope turned to be an oval with semiaxes $R \pm 0.1R$. The dimensions of the envelope are as follows: length 110 mm, radius 52 mm, thickness 0.22 mm. The results of the experiment agree well with the data of [2].

Hence, the applicability of the St. Venant principle in the theory of circular cylindrical envelopes is not doubted only in the case of an axisymmetric stressed-and-strained state caused by the corresponding force and temperature fields. This is also true for other types of envelopes of zero Gaussian curvature, in particular, circular conical envelopes [8]. As concerns noncircular-contour envelopes of zero curvature, for them this unique possibility of applying the St. Venant principle disappears due to the absence of an axisymmetric end effect as such in them.

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NOTATION

R, h, radius and thickness of the envelope; E, ν, modulus of elasticity and Poisson coefficient of the material of the envelope; p_1, p_2, P, intensity of the external force action and the force; t_1, t_2, temperature of the internal and external surfaces of the envelope; t^*, t^{**}, mean temperature over the thickness and temperature difference over the thickness; α_t, coefficient of linear temperature expansion of the material of the envelope; α, β, dimensionless longitudinal and circular coordinates; $2\alpha R_0$, length of the loaded region; ξ_1, ξ_2, boundaries of the loaded or heated region; ξ, coordinate of the line of application of the action concentrated along the generatrix and of the middle of the loaded region; l, length of the envelope; $\mathscr{L}$, differential operator.

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